Benson Example 5.10

Let the primed frame be attached to the wedge. Unprimed frame be the inertial one. If $\vec{X}$ is the trajectory of m lying on the wedge, and $\vec{Z}$ is that of the wedge itself,

$$\ddot{X}_i = \ddot{Z}_i + \ddot{X}_{i'} \tag{0.0.1}$$
$$\ddot{X}_j = \ddot{Z}_j + \ddot{X}_{j'}. \tag{0.0.2}$$

Forces on wedge: normal force $\vec{N} = N\sin\theta\widehat{i} - N\cos\theta\widehat{j}$ exerted by mass m on it, plus the (vertical) normal force exerted on it by the ground. However, vertical forces must cancel out because $\ddot{Z}_j = 0$. Horizontal forces:

$$M\ddot{Z}_i \equiv MA_i = N\sin\theta. \tag{0.0.3}$$

Forces on mass m: normal force $-\vec{N} = -N\sin\theta\widehat{i} + N\cos\theta\widehat{j}$ exerted on it by the wedge, weight $-mg\widehat{j}$.

$$m\ddot{X}_i = m(A_i + \ddot{X}_{i'}) = -N\sin\theta \tag{0.0.4}$$
$$m\ddot{X}_j = m\ddot{X}_{j'} = N\cos\theta - mg. \tag{0.0.5}$$

Note that in the primed frame, $X_{i'}\tan\theta = X_{j'}$; the mass m is constrained to move on the slope of angle θ. Therefore

$$\ddot{X}_{i'}\tan\theta = \ddot{X}_{j'}. \tag{0.0.6}$$

Hence, inserting eq. (0.0.3) into eq. (0.0.4); and equations (0.0.3) and (0.0.6) into eq. (0.0.5):

$$m(A_i + \ddot{X}_{i'}) = -MA_i \tag{0.0.7}$$
$$m\ddot{X}_{i'}\tan\theta = \frac{MA_i}{\sin\theta}\cos\theta - mg. \tag{0.0.8}$$

The rest is algebra to get rid of $\ddot{X}_{i'}$. First line allows us to solve $\ddot{X}_{i'}$ for A_i.

$$(m+M)A_i = -m\ddot{X}_{i'} \tag{0.0.9}$$
$$\ddot{X}_{i'} = -\frac{m+M}{m}A_i. \tag{0.0.10}$$

Then, eq. (0.0.8) becomes

$$-(m+M)A_i\frac{\sin\theta}{\cos\theta} = \frac{MA_i}{\sin\theta}\cos\theta - mg \tag{0.0.11}$$

$$mg = A_i \left(\frac{M}{\sin\theta} \cos\theta + (m+M) \frac{\sin\theta}{\cos\theta} \right) \tag{0.0.12}$$

$$A_i \left(M \cos^2\theta + (m+M) \sin^2\theta \right) = mg \sin\theta \cos\theta \tag{0.0.13}$$

$$A_i \left(M + m \sin^2\theta \right) = mg \sin\theta \cos\theta \tag{0.0.14}$$

$$A_i = \frac{mg \sin\theta \cos\theta}{M + m \sin^2\theta}. \tag{0.0.15}$$

Remember: $\cos^2\theta + \sin^2\theta = 1$ is simply the Pythagorean theorem.