General Physics A: Final

1 Kepler's Laws of Planetary Motion

Gravitational Binary System The key goal of this problem is to derive Kepler's laws of planetary motion using the conservation laws we have learned this semester.

Consider two bodies located at $\vec{x}_1$ and $\vec{x}_2$, with masses m_1 and m_2 respectively. Let us assume they interact solely through the force of gravity, without any external forces. Newton's second law now reads

$$m_1 \frac{\mathrm{d}^2 \vec{x}_1}{\mathrm{d}t^2} = -\frac{G_\mathrm{N} m_1 m_2}{|\vec{x}_1 - \vec{x}_2|^2} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|}, \tag{1.0.1}$$

$$m_2 \frac{\mathrm{d}^2 \vec{x}_2}{\mathrm{d}t^2} = +\frac{G_\mathrm{N} m_1 m_2}{|\vec{x}_1 - \vec{x}_2|^2} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|}. \tag{1.0.2}$$

Galilean Transformation To begin, explain why Newton's laws in equations (1.0.1) and (1.0.2) take the *same form* if we switch from the inertial frame $\{\vec{x}_1, \vec{x}_2\}$ to another inertial frame $\{\vec{x}'_1, \vec{x}'_2\}$ related by

$$\vec{x} = \vec{x}' + \vec{v}_0 (t - t_0); \tag{1.0.3}$$

where $\vec{v}_0$ and t_0 are constants-in-time. The $\vec{v}_0 \cdot t_0$ may be viewed as a constant displacement of the coordinate system; whereas $\vec{v}_0 t$ relates two inertial frames that are moving at a constant velocity $\vec{v}_0$ with respect to each other.

Center-of-Mass Frame Let $\vec{X}_\mathrm{COM}$ be the location of the center-of-mass of the above binary system; and let

$$\vec{\Delta} = \vec{x}_1 - \vec{x}_2. \tag{1.0.4}$$

Show that

$$\ddot{\vec{X}}_\mathrm{COM} = 0, \tag{1.0.5}$$

$$\ddot{\vec{\Delta}} = -\frac{G_\mathrm{N} M}{|\vec{\Delta}|^2} \frac{\vec{\Delta}}{|\vec{\Delta}|}; \tag{1.0.6}$$

where $M = m_1 + m_2$ is the total mass; and each overdot denotes a time derivative.

Energy and Angular Momentum Conservation Explain why the following are constants-in-time:

$$E \equiv \frac{1}{2} \dot{\vec{\Delta}}^2 - \frac{G_\mathrm{N} M}{|\vec{\Delta}|}, \tag{1.0.7}$$

$$\vec{L} \equiv \frac{1}{2} \vec{\Delta} \times \dot{\vec{\Delta}}. \tag{1.0.8}$$

Kepler's 2nd Law Explain why the conservation of eq. (1.0.8) implies the motion described by $\vec{\Delta}$ must always lie on a fixed 2-dimensional (2D) plane. Next, explain why $|\vec{L}| \equiv \dot{A}$ is in fact the area swept out by the $\vec{\Delta}-$motion per unit time. This is Kepler's 2nd law.

Polar Coordinates Since the binary system must move on a fixed plane, we may now use polar coordinates:

$$\vec{\Delta} = (r\cos\phi, r\sin\phi, 0). \tag{1.0.9}$$

Show that eq. (1.0.8) is equivalent to

$$\dot{\phi} = \frac{2\dot{A}}{r^2}; \tag{1.0.10}$$

and eq. (1.0.7) becomes

$$E = \frac{2\dot{A}^2 r'^2}{r^4} - \frac{G_{\rm N} M}{r} + \frac{2\dot{A}^2}{r^2}. \tag{1.0.11}$$

Here, we have re-expressed r in terms of ϕ and have denoted

$$r'(\phi) \equiv \mathrm{d}r/\mathrm{d}\phi. \tag{1.0.12}$$

Kepler's 1st Law Now, an ellipse (x, y) on the 2D plane obeys

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1, \tag{1.0.13}$$

for some *semi-major* axis $a > 0$ and *eccentricity* e subject to the constraint $0 \leq e < 1$. On the other hand, a hyperbola (x, y) on the 2D obeys

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1, \tag{1.0.14}$$

but the eccentricity e is now subject to the constraint $e > 1$. It turns out that both equations (1.0.13) and (1.0.14) may be parametrized by

$$(x, y) = (ae, 0) + r(\phi)(\cos\phi, \sin\phi), \tag{1.0.15}$$

where $(ae, 0)$ is the focus of the ellipse or hyperbola and

$$r(\phi) = \frac{a(1-e^2)}{1 + e\cos\phi}. \tag{1.0.16}$$

(You *do not* need to derive this result.) Insert eq. (1.0.16) into eq. (1.0.11) and solve for a so that E is indeed a constant. With your solution for a, further demonstrate that

$$E = -\frac{1-e^2}{8}\left(\frac{G_{\rm N} M}{\dot{A}}\right)^2. \tag{1.0.17}$$

Is E positive or negative for elliptical orbits? What about for hyperbolic orbits?

Notice you did not have to solve Newton's 2nd law in eq. (1.0.6)! But you have now shown that $r(\phi)$ in fact describes either an ellipse or a hyperbola, provided a yields a constant E.[1] This is the (generalized) Kepler's 1st law.

Kepler's 3rd Law Finally, use the total area of the ellipse

$$A = \pi a^2 \sqrt{1 - e^2} \tag{1.0.18}$$

to deduce Kepler's 3rd law. For elliptical orbits, and with T denoting a period, show that

$$T^2 = \frac{4\pi^2 a^3}{G_{\rm N} M}. \tag{1.0.19}$$

[1] You are given the form of the solution in eq. (1.0.16); extra credit if you can *derive* it by integrating eq. (1.0.11) directly.