Linear and Angular Momentum

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1 Linear Momentum

1.1 Definition and Conservation

A body of mass m and velocity $\vec{v} \equiv \mathrm{d}\vec{x}/\mathrm{d}t$ (with $\vec{x}$ being its displacement vector) is defined to have linear momentum

$$\vec{p} \equiv m\vec{v}. \tag{1.1.1}$$

Newton's second law now reads

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F}. \tag{1.1.2}$$

Kinetic energy is

$$\frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\left(\frac{\vec{p}}{m}\right)^2 = \frac{\vec{p}^2}{2m}. \tag{1.1.3}$$

If U denotes the potential energy, total mechanical energy is

$$E = \frac{\vec{p}^2}{2m} + U. \tag{1.1.4}$$

Conservation of linear momentum Whether or not mechanical energy is conserved, linear momentum of an isolated system is *always* conserved as long as Newton's 2nd and 3rd laws hold. This is the key reason for defining linear momentum in the first place.

Let us see why. Consider the time derivative of the total linear momentum of N bodies

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N}\vec{p}_i = \sum_{i=1}^{N}\vec{F}_i, \tag{1.1.5}$$

where $\vec{F}_i$ is the total force asserted on the i-th body. But since we are assuming these N bodies are isolated, this force must be due to the rest of the $N-1$ bodies:

$$\vec{F}_i = \sum_{j\neq i}^{N}\left(\vec{F} \text{ on } i\text{-th body due to } j\text{-th body}\right). \tag{1.1.6}$$

By Newton's 3rd law, we have

$$\vec{F} \text{ on } i\text{-th body due to } j\text{-th body} = -\left(\vec{F} \text{ on } j\text{-th body due to } i\text{-th body}\right), \tag{1.1.7}$$

for $i \neq j$. If we further define

$$\vec{F} \text{ on } i\text{-th body due to } j\text{-th body} \equiv \vec{F}_{i,j}; \tag{1.1.8}$$

then

$$\vec{F}_{i,j} = -\vec{F}_{j,i}, \tag{1.1.9}$$

$$\vec{F}_i = \sum_{j\neq i}^{N}\vec{F}_{i,j}. \tag{1.1.10}$$

At this point, we return to the total time derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N}\vec{p}_i = \sum_{\substack{1\leq i,j\leq N\\ i\neq j}}\vec{F}_{i,j}. \tag{1.1.11}$$

We are summing over all pairs (i,j) for $i \neq j$. Moreover, for a fixed pair (i,j), we may identify a corresponding pair (j,i). But since $F_{i,j} = -\vec{F}_{j,i}$ these internal forces would cancel in pairs by Newton's 3rd law, resulting in zero net force. This means

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N}\vec{p}_i = 0 \tag{1.1.12}$$

and

$$\sum_{i=1}^{N}\vec{p}_i = \text{constant in time}. \tag{1.1.13}$$

1.2 Center of Mass

A closely related concept is that of the center-of-mass. It is the position $\vec{X}_{\text{COM}}$ within a system that amounts to a 'weighted average' of the positions of the system's individual components. Specifically, for a system of N bodies

$$\vec{X}_{\text{COM}} \equiv \frac{\sum_{1 \leq i \leq N} m_i \vec{x}_i}{\sum_{j=1}^{N} m_j}. \tag{1.2.1}$$

Let us observe that the total linear momentum is simply the total mass times $\mathrm{d}\vec{X}_{\text{COM}}/\mathrm{d}t$:

$$\vec{P}_{\text{COM}} = \left(\sum_{j=1}^{N} m_j\right) \frac{\mathrm{d}\vec{X}_{\text{COM}}}{\mathrm{d}t} \equiv \left(\sum_{j=1}^{N} m_j\right) \vec{v}_{\text{COM}} \tag{1.2.2}$$

$$= \sum_{1 \leq i \leq N} m_i \frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t} = \sum_{1 \leq i \leq N} \vec{p}_i. \tag{1.2.3}$$

That means for an isolated system – where there are no external forces – the center-of-mass must move at a constant velocity:

$$\frac{\mathrm{d}^2 \vec{X}_{\text{COM}}}{\mathrm{d}t^2} = \left(\sum_{j=1}^{N} m_j\right)^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \sum_{1 \leq i \leq N} \vec{p}_i = 0. \tag{1.2.4}$$

At this point, let us define the Galilean transformation

$$\vec{z}_i \equiv \vec{x}_i - \vec{v}_{\text{COM}} \cdot t, \tag{1.2.5}$$

$$\vec{v}_{\text{COM}} \equiv \frac{\mathrm{d}\vec{X}_{\text{COM}}}{\mathrm{d}t}; \tag{1.2.6}$$

and the momenta

$$\vec{q}_i \equiv m_i \dot{\vec{z}}_i \equiv \vec{p}_i - m_i \vec{v}_{\text{COM}}. \tag{1.2.7}$$

The total linear momentum in the new $\vec{z}-$frame is zero:

$$\vec{Q} \equiv \sum_i \vec{q}_i = \sum_i \vec{p}_i - \left(\sum_i m_i\right) \vec{v}_{\text{COM}} = \vec{0}. \tag{1.2.8}$$

We have arrived at the key fact:

> By performing a Galilean transformation to switch into a center-of-mass inertial frame, the total linear momentum of an isolated system can always be set to zero.

Composite Bodies Suppose we have various extended bodies whose COMs we have already computed; and we place these bodies together to form a larger system – what is the COM of this larger system? Let $\vec{x}_{A,i}$ and $m_{A,i}$ be respectively the position and mass of the ith constituent of the Ath body, so that the COM of this Ath body is

$$\vec{X}_{\text{COM},A} = \frac{\sum_i m_{A,i}}{M_A}, \tag{1.2.9}$$

$$M_A \equiv \sum_j m_{A,j}. \tag{1.2.10}$$

On the other hand, the COM of the entire system involves a sum over both i and A because we now need to include every constituent:

$$\vec{X}_{\text{COM}} = \frac{\sum_{i,A} m_{A,i}\vec{x}_{A,i}}{\sum_{j,B} m_{B,j}} = \frac{\sum_{i,A} m_{A,i}\vec{x}_{A,i}}{\sum_B M_B}. \tag{1.2.11}$$

However, for a fixed A, we may multiply and divide by the total mass of the Ath body M_A:

$$\vec{X}_{\text{COM}} = \sum_A \left(\frac{M_A}{\sum_B M_B} \frac{\sum_i m_{A,i}\vec{x}_{A,i}}{M_A} \right) \tag{1.2.12}$$

$$= \frac{\sum_A M_A \vec{X}_{\text{COM},A}}{\sum_B M_B}. \tag{1.2.13}$$

We infer:

> The COM of a system comprised of $N \geq 2$ bodies can be computed by treating these N bodies as point masses centered at their individual COMs.

1.3 Kinetic Energy

For an arbitrary system let the displacement vector of the i-th body be decomposed into $\vec{X}_{\text{COM}}$ plus the displacement $\vec{X}_i$ from the COM to it; i.e.,

$$\vec{x}_i \equiv \vec{X}_{\text{COM}} + \vec{X}_i. \tag{1.3.1}$$

Note that, by the very definition of COM, we have

$$\vec{X}_{\text{COM}} = \frac{\sum_i m_i \vec{x}_i}{M_{\text{total}}} = \frac{\sum_i m_i (\vec{X}_{\text{COM}} + \vec{X}_i)}{M_{\text{total}}} = \vec{X}_{\text{COM}} + \frac{\sum_i m_i \vec{X}_i}{M_{\text{total}}}. \tag{1.3.2}$$

This implies

$$\sum_i m_i \vec{X}_i = \vec{0}. \tag{1.3.3}$$

Then, the total KE is

$$\sum_i \frac{m_i}{2}\left(\vec{v}_{\text{COM}}^2 + \dot{\vec{X}}_i^2 + 2\vec{v}_{\text{COM}} \cdot \dot{\vec{X}}_i \right) \tag{1.3.4}$$

$$= \frac{M_{\text{total}}}{2}\vec{v}_{\text{COM}}^2 + \sum_i \frac{m_i}{2}\dot{\vec{X}}_i^2 + M_{\text{total}}\vec{v}_{\text{COM}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\frac{\sum_i m_i \vec{X}_i}{M_{\text{total}}} \tag{1.3.5}$$

$$= \frac{M_{\text{total}}}{2}\vec{v}_{\text{COM}}^2 + \sum_i \frac{m_i}{2}\dot{\vec{X}}_i^2 \tag{1.3.6}$$

$$= \text{KE}_{\text{COM}} + \text{KE relative to COM}. \tag{1.3.7}$$

1.4 Collisions

One application of the conservation of linear momentum is to that of collisions of N bodies. Let us denote the momentum of the incoming bodies to be $\{\vec{p}_1, \dots, \vec{p}_N\}$ and those of the outgoing bodies to be $\{\vec{p}'_1, \dots, \vec{p}'_N\}$ Then, conservation of linear momentum says

$$\sum_{i=1}^{N} \vec{p}_i = \sum_{i=1}^{N} \vec{p}'_i. \tag{1.4.1}$$

If we are operating in the center-of-mass frame,

$$\sum_{i=1}^{N} \vec{q}_i = \vec{0} = \sum_{i=1}^{N} \vec{q}'_i. \tag{1.4.2}$$

If total mechanical energy is conserved,

$$\sum_{i=1}^{N} \left(\frac{\vec{p}_i^2}{2m_i} + U_i \right) = \sum_{i=1}^{N} \left(\frac{\vec{p}_i'^2}{2m_i} + U'_i \right), \tag{1.4.3}$$

where the primes on the right hand side simply denotes 'outgoing'.

2 Body Collision: Elastic Case For the 2−body problem, and in the center-of-mass frame, we must be $\vec{q}_1 = -\vec{q}_2$ and $\vec{q}'_1 = -\vec{q}'_2$. Let us place the $z-$axis along the incoming momentum:

$$\vec{q}_1 = (0, 0, q), \tag{1.4.4}$$
$$\vec{q}_2 = (0, 0, -q); \tag{1.4.5}$$

and the outgoing momentum must read

$$\vec{q}'_1 = q'(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{1.4.6}$$
$$\vec{q}'_2 = -q'(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{1.4.7}$$

Suppose we may neglect the potential energies before and after the collisions. Then, if total mechanical energy is conserved,

$$q^2 \left(\frac{1}{2m_1} + \frac{1}{2m_2} \right) = q'^2 \left(\frac{1}{2m_1} + \frac{1}{2m_2} \right). \tag{1.4.8}$$

In other words, the magnitudes of the momentum also needs to be conserved *if* mechanical energy is conserved:

$$q = q'. \tag{1.4.9}$$

There are no further constraints on the outgoing angles (θ, ϕ) *unless* we have further details regarding how the two bodies interacted. In *quantum mechanics* even if the detailed interactions were specified, the best one can do is to calculate the probability (density) that a given outgoing direction (θ, ϕ) would occur.

Head-On For the head-on collision, we have

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \tag{1.4.10}$$
$$m_1(v_1 - v_1') = m_2(v_2' - v_2). \tag{1.4.11}$$

(Twice of) Energy conservation says

$$m_1 v_1^2 + m_2 v_2^2 = m_1 v_1'^2 + m_2 v_2'^2, \tag{1.4.12}$$
$$m_1(v_1 - v_1')(v_1 + v_1') = m_2(v_2' - v_2)(v_2' + v_2). \tag{1.4.13}$$

Recalling the momentum conservation equations,

$$v_1 - v_2 = v_2' - v_1'. \tag{1.4.14}$$

We may express

$$\begin{bmatrix} m_1 & m_2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix}. \tag{1.4.15}$$

The inverse of a 2×2 matrix is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1.4.16}$$

That means the outgoing velocities may be expressed in terms of the incoming ones:

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = (m_1 + m_2)^{-1} \begin{bmatrix} 1 & -m_2 \\ 1 & m_1 \end{bmatrix} \begin{bmatrix} m_1 & m_2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{1.4.17}$$
$$= (m_1 + m_2)^{-1} \begin{bmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{1.4.18}$$

If, for instance, m_2 were initially at rest, $v_2 = 0$ and

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = v_1 (m_1 + m_2)^{-1} \begin{bmatrix} m_1 - m_2 \\ 2m_1 \end{bmatrix}. \tag{1.4.19}$$

If $m_1/m_2 \to 0$, namely if m_1 is much lighter than the target initially at rest,

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} \approx v_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \tag{1.4.20}$$

On the other hand, if $m_2/m_1 \to 0$, the target is much lighter than the incident mass,

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \tag{1.4.21}$$

2 Body Collision: Inelastic Case If mechanical energy (really kinetic) is not conserved, then we can still work in the center of mass frame but $q \neq q'$. Suppose we consider the perfectly inelastic case where the two bodies stick together after the collision – the resulting mass of $m_1 + m_2$ is then described by a single momentum $\vec{q}'$ – we must have

$$\vec{q}' = 0. \tag{1.4.22}$$

This in turn means the final kinetic energy is zero in the center of mass frame; i.e., the initial kinetic energy $(q^2/2)(m_1^{-1} + m_2^{-1})$ is completely lost.

1.5 External Forces

If there is an external force $\vec{F}_{i,\text{ext}}$ on the i-th body in the system, then running the argument through once again will tell us the time derivative of the total momentum would now involve the sum of internal and external forces. However, the internal forces still cancel, by Newton's third law. Hence, what remains is the total external force.

$$M_{\text{total}} \frac{\mathrm{d}^2 \vec{X}_{\text{COM}}}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_i \vec{p}_i = \sum_i \vec{F}_{i,\text{ext}} \equiv \vec{F}_{\text{total}}, \tag{1.5.1}$$

$$M_{\text{total}} \equiv \sum_i m_i. \tag{1.5.2}$$

2 Angular Momentum

The notion of angular momentum is particularly important when solving problems involving rotation. For a mass m with displacement $\vec{x}$ and momentum $\vec{p}$, its angular momentum is defined as

$$\vec{L} \equiv \vec{x} \times \vec{p} = m\vec{x} \times \dot{\vec{x}}. \tag{2.0.1}$$

Notice this definition depends on the choice of origin. That is, under a shift

$$\vec{x} \to \vec{x} + \vec{x}_0, \tag{2.0.2}$$

we have

$$\vec{L} \to \vec{L} + \vec{x}_0 \times \vec{p}. \tag{2.0.3}$$

Now suppose our system comprises of N bodies. We may now consider the time derivative of its total angular momentum

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{N} (\vec{x}_i \times \vec{p}_i) = \sum_{i=1}^{N} \left(\frac{\vec{p}_i}{m} \times \vec{p}_i \right) + \sum_{i=1}^{N} (\vec{x}_i \times \vec{F}_i) \tag{2.0.4}$$

$$= \sum_{i=1}^{N} (\vec{x}_i \times \vec{F}_i). \tag{2.0.5}$$

We may define torque as

$$\vec{\tau} \equiv \vec{x} \times \vec{F} \tag{2.0.6}$$

– which, like angular momentum, depends on the choice of the origin – and therefore summarize:

The time derivative of the total angular momentum is equal to the total torque.

This is the analog of: the time derivative of the total linear momentum is equal to the total force.

Conservation of angular momentum We may also immediately deduce:

The angular momentum of a system is conserved whenever the total torque applied is zero.

Radial forces A special case worth highlighting is that of a radial force. A body of mass m subject to a radial force $\vec{F} = -V'(r)\widehat{r}$ experiences zero torque as computed about the origin because

$$\vec{\tau} = \vec{x} \times \vec{F} \tag{2.0.7}$$
$$= -rV'(r)\widehat{r} \times \widehat{r} = 0. \tag{2.0.8}$$

In other words, sometimes the total force is non-zero but the total torque is zero. Obviously, if the total force is zero, there is no torque to speak of.

Kinetic energy of rigid rotating body By rigid body we mean that the distance between *any* pair of points within the body must remain the same at all times. What this means is that *internal* forces are strong enough to hold the body together, preserving its shape, size, etc. Let us choose to choose the coordinate system such that the ith constituent is located at

$$\vec{x}_i = \vec{X}_{\text{COM}} + \vec{X}_i. \tag{2.0.9}$$

Let us suppose the body is rotating along the $\widehat{k}$ axis, which we shall assume is passing through the COM, and let us parametrize

$$\vec{X}_i = r_i \widehat{r}_i + z_i \widehat{k}. \tag{2.0.10}$$

We must have z_i remaining constant, since we are assuming the body is rotating about the $\widehat{k}$ axis. But rigid body means $|\vec{X}_i| = \sqrt{z_i^2 + r_i^2}$ is constant; and therefore r_i is constant too. Recalling $\dot{\widehat{r}} = \dot{\theta}\widehat{\theta}$,

$$\dot{\vec{X}}_i = \omega r_i \widehat{\theta}_i, \tag{2.0.11}$$
$$\omega \equiv \dot{\theta}. \tag{2.0.12}$$

Because this is a rigid body, ω is common to each and every ith constituent; i.e., every piece of the body rotates at the same rate – otherwise the body would begin to deform. The total KE, according to eq. (1.3.6), now becomes

$$\text{Total KE} = \frac{M_{\text{total}}}{2}\vec{v}_{\text{COM}}^2 + \frac{1}{2} I_{\text{COM}} \omega^2, \tag{2.0.13}$$
$$I_{\text{COM}} \equiv \sum_i m_i r_i^2; \tag{2.0.14}$$

where I_{COM} is known as the moment of inertia, here computed about the COM.

Angular Velocity For fixed axis rotation, we should really denote angular velocity as a vector:

$$\vec{\omega} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{k} = \omega \widehat{k}, \tag{2.0.15}$$

where $\widehat{k}$ is the axis of rotation. We may also discover that

$$\begin{aligned} \widehat{\omega} \times \vec{X}_i &= \omega \widehat{k} \times \left(r_i \widehat{r}_i + z_i \widehat{k} \right) \\ &= \omega r_i \widehat{\theta}_i = \frac{\mathrm{d}\vec{X}_i}{\mathrm{d}t}. \end{aligned} \tag{2.0.16}$$

Additionally,

$$\vec{\omega} \times \left(\vec{\omega} \times \vec{X}_i \right) = \omega^2 r_i \widehat{k} \times \widehat{\theta}_i = -\omega^2 r_i \widehat{r}_i. \tag{2.0.17}$$

Denoting the second time derivative of the angle as

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \frac{\mathrm{d}\omega}{\mathrm{d}t} \equiv \alpha, \tag{2.0.18}$$

we may also see that

$$\vec{\alpha} \equiv \alpha \widehat{k}, \tag{2.0.19}$$

$$\vec{\alpha} \times \vec{X}_i = \alpha r_i \widehat{\theta}_i. \tag{2.0.20}$$

Employing $\dot{\widehat{\theta}} = -\omega \widehat{r}$, we see the acceleration of the displacement from the COM is

$$\ddot{\vec{X}}_i = \alpha r_i \widehat{\theta}_i - \omega^2 r_i \widehat{r}_i \tag{2.0.21}$$

$$= \vec{\alpha} \times \vec{X}_i + \vec{\omega} \times \left(\vec{\omega} \times \vec{X}_i \right). \tag{2.0.22}$$

Parallel Axis Theorem Suppose, instead of computing I about an axis passing through the COM, we compute it about an axis parallel to it – namely, choose $\vec{X}_0$ such that

$$\vec{X}_0 = \vec{X}_{\mathrm{COM}} + r_0 \widehat{n}, \tag{2.0.23}$$

where the unit vector $\widehat{n}$ is perpendicular to the rotation axis. If we decompose

$$\vec{X}_{\mathrm{COM}} = \vec{X}^{\perp}_{\mathrm{COM}} + z_{\mathrm{COM}} \widehat{k}, \tag{2.0.24}$$

$$\vec{X}_0 = \vec{X}^{\perp}_0 + z_0 \widehat{k}, \tag{2.0.25}$$

where $\vec{X}^{\perp}_{\mathrm{COM}}$ and $\vec{X}^{\perp}_0$ lie on the plane perpendicular to the rotation axis $\widehat{k}$, we may deduce

$$r_i'^2 \equiv (\vec{X}^{\perp}_0 - r_i \widehat{r}_i)^2 = (\vec{X}^{\perp}_{\mathrm{COM}} + r_0 \widehat{n} - r_i \widehat{r}_i)^2 \tag{2.0.26}$$

$$= (\vec{X}^{\perp}_{\mathrm{COM}} - r_i \widehat{r}_i)^2 + r_0^2 + 2 r_0 \widehat{n} \cdot (\vec{X}^{\perp}_{\mathrm{COM}} - r_i \widehat{r}_i). \tag{2.0.27}$$

The momentum of inertia computed about an axis parallel to the one passing through the COM at a distance r_0 away, is

$$I = \sum_i m_i r_i'^2 = \sum_i m_i r_i^2 + M_{\mathrm{total}} r_0^2 + 2 r_0 \widehat{n} \sum_i m_i (\vec{X}^{\perp}_{\mathrm{COM}} - r_i \widehat{r}_i) \tag{2.0.28}$$

$$= I_{\mathrm{COM}} + M_{\mathrm{total}} r_0^2. \tag{2.0.29}$$

Newton's Laws for rigid body We may compute the total angular momentum.

$$\vec{L}_{\text{total}} = \sum_i m_i(\vec{X}_{\text{COM}} + \vec{X}_i) \times (\dot{\vec{X}}_{\text{COM}} + \dot{\vec{X}}_i) \tag{2.0.30}$$

$$= \sum_i m_i \left(\vec{X}_{\text{COM}} \times \dot{\vec{X}}_{\text{COM}} + \vec{X}_{\text{COM}} \times \dot{\vec{X}}_i + \vec{X}_i \times \dot{\vec{X}}_{\text{COM}} + \vec{X}_i \times \dot{\vec{X}}_i\right). \tag{2.0.31}$$

We proved earlier that, if $\vec{X}_i$ is the displacement from the COM, then $\sum_i m_i\vec{X}_i = \vec{0}$. Therefore $\sum_i m_i\dot{\vec{X}}_i = \vec{0}$ too.

$$\vec{L}_{\text{total}} = M_{\text{total}}\vec{X}_{\text{COM}} \times \dot{\vec{X}}_{\text{COM}} + \sum_i m_i\vec{X}_i \times \dot{\vec{X}}_i \tag{2.0.32}$$

The first term can be thought of as the angular momentum of a point mass with the total mass of the system located at its COM. Again, if the body is rotating along the $\widehat{k}$ axis,

$$\sum_i m_i\vec{X}_i \times \dot{\vec{X}}_i = \omega \sum_i m_i r_i(r_i\widehat{r}_i + z_i\widehat{k}) \times \widehat{\theta}_i \tag{2.0.33}$$

$$= \omega I_{\text{COM}}\widehat{k} - \omega \sum_i m_i r_i z_i \widehat{r}_i. \tag{2.0.34}$$

If we assume the sum of the $m_i r_i \widehat{r}_i$ is zero for a fixed $z-$slice (i.e., for all z_i taking the same value) then

$$\vec{L}_{\text{total}} = I_{\text{COM}}\vec{\omega}. \tag{2.0.35}$$

We have already derived, the time derivative of the angular momentum is equal to the total torque $\vec{\tau} = \tau\widehat{k}$:

$$I_{\text{COM}}\vec{\alpha} = \vec{\tau}, \tag{2.0.36}$$

where $\vec{\alpha} = \alpha\widehat{k} \equiv \dot{\omega}\widehat{k} = \ddot{\theta}\widehat{k}$. Note that many assumptions were made – i.e., this result holds because we have assumed a rigid body undergoing fixed axis rotations.

Work-Energy Theorem for rigid bodies If we assume the rigid body is constrained to rotate along the $\widehat{k}$ axis, then if we decompose all vectors as

$$\vec{x}_i = \vec{X}_{\text{COM}} + \vec{X}_i, \tag{2.0.37}$$

where r_i and Z_i are constant in time; we have

$$\mathrm{d}W = \sum_i \vec{F}_i \cdot \mathrm{d}\vec{X}_{\text{COM}} + \sum_i \vec{F}_i \cdot \mathrm{d}\vec{X}_i. \tag{2.0.38}$$

The first term on the right hand side is simply the total force dotted into $\mathrm{d}\vec{X}_{\text{COM}}$; this means it is simply the total work done on the mass M_{total} located at the COM. For the second term, we may use Newton's 2nd law $\vec{F}_i = m_i\ddot{\vec{X}}_i$ and $\mathrm{d}\vec{X}_i = \dot{\vec{X}}_i\mathrm{d}t$. Utilizing equations (2.0.16) and (2.0.21),

$$\mathrm{d}W = \vec{F}_{\text{total}} \cdot \mathrm{d}\vec{X}_{\text{COM}} + \sum_i m_i \left(\vec{\alpha} \times \vec{X}_i + \vec{\omega} \times \left(\vec{\omega} \times \vec{X}_i\right)\right) \cdot (\vec{\omega} \times \vec{X}_i)\mathrm{d}t. \tag{2.0.39}$$

Now, $\vec{\omega} \times (\vec{\omega} \times \vec{X}_i)$ must be perpendicular to $\vec{\omega} \times \vec{X}_i$; i.e., the rightmost term is zero. We may consider

$$\begin{aligned}(\vec{\omega} \times \vec{X}_i) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\vec{\omega} \times \vec{X}_i) &= (\vec{\omega} \times \vec{X}_i) \cdot \Big((\vec{\alpha} \times \vec{X}_i) + \vec{\omega} \times (\vec{\omega} \times \vec{X}_i)\Big) &(2.0.40)\\ &= (\vec{\omega} \times \vec{X}_i) \cdot (\vec{\alpha} \times \vec{X}_i). &(2.0.41)\end{aligned}$$

Also remember,

$$(\vec{\omega} \times \vec{X}_i)^2 = \omega r_i \widehat{\theta}_i. \tag{2.0.42}$$

At this point, we have

$$\begin{aligned}\mathrm{d}W &= \vec{F}_{\text{total}} \cdot \mathrm{d}\vec{X}_{\text{COM}} + \sum_i \frac{m_i}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\omega} \times \vec{X}_i\right)^2 \mathrm{d}t &(2.0.43)\\ &= M_{\text{total}}\frac{\mathrm{d}^2\vec{X}_{\text{COM}}}{\mathrm{d}t^2} \cdot \frac{\mathrm{d}\vec{X}_{\text{COM}}}{\mathrm{d}t}\mathrm{d}t + \mathrm{d}t\frac{\mathrm{d}}{\mathrm{d}t}\sum_i \omega^2\frac{m_i r_i^2}{2} &(2.0.44)\\ &= \mathrm{d}t\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{M_{\text{total}}}{2}\left(\frac{\mathrm{d}\vec{X}_{\text{COM}}}{\mathrm{d}t}\right)^2 + \frac{I_{\text{COM}}\omega^2}{2}\right). &(2.0.45)\end{aligned}$$

Total work done may now be deduced to be

$$W = \Delta\left(\frac{M_{\text{total}}}{2}\left(\frac{\mathrm{d}\vec{X}_{\text{COM}}}{\mathrm{d}t}\right)^2 + \frac{I_{\text{COM}}\omega^2}{2}\right) = \Delta\text{KE}. \tag{2.0.46}$$

Moreover, since $\vec{\tau} = I_{\text{COM}}\alpha\widehat{k}$, we may also express

$$\mathrm{d}t\frac{\mathrm{d}}{\mathrm{d}t}\frac{I_{\text{COM}}\omega^2}{2} = \mathrm{d}t\frac{\mathrm{d}\theta}{\mathrm{d}t}I_{\text{COM}}\alpha = \mathrm{d}\theta\tau; \tag{2.0.47}$$

and therefore

$$\int_{\theta_{\text{initial}}}^{\theta_{\text{final}}} \tau\mathrm{d}\theta = \Delta\left(\frac{I_{\text{COM}}\omega^2}{2}\right). \tag{2.0.48}$$