General Physics A: Midterm 1

1 Rotating frames are not inertial

In the absence of forces, a body will maintain a constant velocity when viewed by observers in an inertial frame. Hence, if there are no forces but yet the body is accelerating, the reference frame must be non-inertial. In this problem you will explore the example of a rotating frame.

Consider a flat circular disk of radius R rotating in a counterclockwise manner at angular frequency ω. Define a Cartesian coordinate system of an inertial frame, with origin placed at the center of the disk, so that a given point on the disk can be described by

$$\vec{X} = r\cos(\theta_R + \omega t)\widehat{i} + r\sin(\theta_R + \omega t)\widehat{j}. \tag{1.0.1}$$

Now attach Cartesian axes $\widehat{i}_R$ and $\widehat{j}_R$ to this rotating disk, with its origin also placed at its center, such that $\widehat{i} = \widehat{i}_R$ and $\widehat{j} = \widehat{j}_R$ at $t = 0$. That is, these $\widehat{i}_R$ and $\widehat{j}_R$ form the axes of our rotating frame and they coincide with the inertial frame axes at $t = 0$.

Let Bob sit on the rotating disk so that at $t = 0$ he is sitting at $-(R/2)\widehat{i}$ and let Alice sit on the rotating disk so that at $t = 0$ she is at $(R/2)\widehat{i}$; namely, they are sitting directly opposite each other with respect to the center of the disk. Suppose, at $t = 0$, Bob rolls a ball which then hits Alice at time $t = T > 0$. Show that in the rotating frame of Bob and Alice and for $0 \le t \le T$, the trajectory is given by

$$\begin{aligned}\vec{X}_R = \frac{R}{2T}\left((t-T)\cos(\omega t) + t\cos(\omega(t-T))\right)\widehat{i}_R \\ - \frac{R}{2T}\left((t-T)\sin(\omega t) + t\sin(\omega(t-T))\right)\widehat{j}_R.\end{aligned} \tag{1.0.2}$$

This is clearly not a constant velocity motion! You may assume there are no forces acting on the ball parallel to the surface of the disk itself.

What is the initial velocity of the ball in the rotating frame of Bob and Alice? What is the initial velocity in the inertial frame?

2 Fictitious forces in a rotating frame

In this problem we shall explore the presence of 'fictitious' forces in a rotating frame, by exploiting the polar coordinate system.

In the inertial frame of Problem 1, first show that acceleration can be written as

$$\frac{\mathrm{d}^2\vec{X}}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right)\widehat{r} + \left(2\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t} + r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right)\widehat{\theta}, \tag{2.0.1}$$

where $\widehat{r}$ is the unit radial vector and $\widehat{\theta}$ is the unit azimuthal one.

Next, explain why switching to the rotating frame of Alice and Bob in Problem 1 amounts to the Galilean-like shift

$$\theta_R = \theta - \omega t. \tag{2.0.2}$$

If $\vec{F}_{\text{ext}}$ denotes external forces, by massaging Newton's second law $m\mathrm{d}^2\vec{X}/\mathrm{d}t^2 = \vec{F}_{\text{ext}}$ into

$$m\frac{\mathrm{d}^2\vec{X}_R}{\mathrm{d}t^2} = \vec{F}_{\text{ext}} + \vec{F}_{\text{fic}}, \tag{2.0.3}$$

identify the fictitious force to be

$$\vec{F}_{\text{fic}} = m\left(\omega^2 r + 2\omega r\frac{\mathrm{d}\theta_R}{\mathrm{d}t}\right)\widehat{r} - 2m\omega\frac{\mathrm{d}r}{\mathrm{d}t}\widehat{\theta}. \tag{2.0.4}$$

Hint: How are the radial and angular coordinates in the inertial frame related to those in the rotating frame? $\square$