Work and Energy

Work Let $\vec{F}$ denote some force acting on a body of mass m. The work W done by $\vec{F}$ as m moves from point A to point B along some path P is defined as

$$W \equiv \int_A^B \vec{F}(\vec{x}) \cdot \mathrm{d}\vec{x} = \int_A^B \vec{F} \cdot \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \mathrm{d}t. \tag{0.0.1}$$

Power At a given time t, we may express the integral as

$$W(t) = \int_{t_A}^{t} \vec{F}(\vec{x}(t')) \cdot \frac{\mathrm{d}\vec{x}}{\mathrm{d}t'} \mathrm{d}t', \tag{0.0.2}$$

where we are now viewing the trajectory as a function of time, described by $\vec{x}(t_A \leq t' \leq t)$. Differentiating both sides with respect to time, and recognizing differentiation as the inverse operation of integration,

$$P(t) \equiv \frac{\mathrm{d}W(t)}{\mathrm{d}t} = \vec{F}(\vec{x}(t)) \cdot \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}. \tag{0.0.3}$$

We define *power* $P(t)$ to be the time derivative of work done.

Work-Energy Theorem We now prove

> The total work done on a mass m is equal to the change in its kinetic energy $(1/2)m\vec{v}^2 = (m/2)\dot{\vec{x}} \cdot \dot{\vec{x}}$.

Provided Newton's law $\vec{F} = m\mathrm{d}^2\vec{x}/\mathrm{d}t^2$ holds, and denoting $\mathrm{d}^2\vec{x}/\mathrm{d}t^2 \equiv \ddot{\vec{x}}$ and $\mathrm{d}\vec{x}/\mathrm{d}t \equiv \dot{\vec{x}}$,

$$W_{\text{total}} = \int_A^B \vec{F}_{\text{total}}(\vec{x}) \cdot \mathrm{d}\vec{x} \tag{0.0.4}$$

$$= m \int_A^B \ddot{\vec{x}} \cdot \dot{\vec{x}} \mathrm{d}t = \int_A^B \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{m}{2} \dot{\vec{x}} \cdot \dot{\vec{x}} \right) \mathrm{d}t \tag{0.0.5}$$

$$= \frac{m}{2}\vec{v}^2(B) - \frac{m}{2}\vec{v}^2(A) \equiv \Delta\text{KE}. \tag{0.0.6}$$

Conservative vs non-conservative forces A conservative force $\vec{F}$ is one where it can be written as the negative gradient of a potential energy $U(\vec{x})$,

$$\vec{F} = -\vec{\nabla} U. \tag{0.0.7}$$

In Cartesian coordinates (x, y, z),

$$F_x = -\partial_x U, \qquad F_y = -\partial_y U, \qquad F_z = -\partial_z U. \tag{0.0.8}$$

Note that U is only defined up to a space-independent constant, since any such term would be eliminated by the derivative operation to return the same force $\vec{F}$; namely, $\vec{F} = -\vec{\nabla}(U + \text{constant}) = -\vec{\nabla}U$.

A non-conservative force is simply one that *cannot* be written as a negative gradient of a potential energy – friction is a key example. The key property of conservative forces is that the work done by them is *independent* of the path taken:

$$W = \int_A^B (-\vec{\nabla}U(\vec{x})) \cdot \mathrm{d}\vec{x} \tag{0.0.9}$$

$$= -\int_A^B (\partial_x U \mathrm{d}x + \partial_y U \mathrm{d}y + \partial_z U \mathrm{d}z) \tag{0.0.10}$$

$$= -\int_A^B \mathrm{d}U = U(A) - U(B) \tag{0.0.11}$$

– the work done by a conservative force is equal to the difference between the potential energies at the end points. In fact, the logic goes in reverse too: if the work done by a force $\vec{F}$ is *always* path independent, then it can be expressed as a negative gradient of a potential energy.

Suppose $\{\vec{F}_{1,\text{NC}}, \vec{F}_{2,\text{NC}}, \dots\}$ are non-conservative forces and $\{\vec{F}_{1,\text{C}} = -\vec{\nabla}U_1, \vec{F}_{2,\text{C}} = -\vec{\nabla}U_2, \dots\}$ are conservative ones, then the total work done is

$$\sum_i W_{i,\text{NC}} + \sum_i \int_A^B \vec{F}_{i,\text{C}} \cdot \mathrm{d}\vec{x} = \frac{m\vec{v}^2(B)}{2} - \frac{m\vec{v}^2(A)}{2} \tag{0.0.12}$$

$$\sum_i W_{i,\text{NC}} + \sum_i (U_i(A) - U_i(B)) = \frac{m\vec{v}^2(B)}{2} - \frac{m\vec{v}^2(A)}{2}. \tag{0.0.13}$$

If we define total mechanical energy as

$$E \equiv \frac{m\vec{v}^2}{2} + \sum_i U_i, \tag{0.0.14}$$

this relation can be summarized as:

$$\sum_i W_{i,\text{NC}} = E(B) - E(A) \tag{0.0.15}$$

– the total work done by the *non-conservative* forces acting on m is equal to the change in total mechanical energy arising from kinetic energy plus the conservative-forces' potential energies. In particular, if only conservative forces are present, total mechanical energy is conserved:

$$E(A) = E(B) = \text{constant}. \tag{0.0.16}$$

Gravity Near Earth's Surface Set up a coordinate system such that $\widehat{j}$ is the unit vector perpendicular to and pointing away from the surface of the Earth; then the force of gravity on a mass m is

$$-mg\widehat{j} = -\partial_y(mgy)\widehat{j}, \tag{0.0.17}$$

where the coordinate in the vertical direction. The gravitational potential is therefore

$$U = mgy + \text{constant}. \tag{0.0.18}$$

Gravity: General Case In general, the force of gravity of mass (point) mass M upon another (point) mass m is

$$-\frac{G_{\text{N}}Mm}{r^2}\widehat{r}, \tag{0.0.19}$$

where $\widehat{r}$ is the unit vector that points away from M. If we erect a Cartesian coordinate system centered at M,

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{0.0.20}$$

$$-(\partial_x, \partial_y, \partial_z)r^{-1} = -(\partial_x, \partial_y, \partial_z)(x^2 + y^2 + z^2)^{-1/2} \tag{0.0.21}$$

$$= \frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x, 2y, 2z) \tag{0.0.22}$$

$$= \frac{(x, y, z)}{r^3} = \vec{x}/r^3 = \widehat{r}/r^2. \tag{0.0.23}$$

Therefore, the force of gravity by M on m can now be recognized as

$$-\frac{G_{\text{N}}Mm}{r^2}\widehat{r} = -\vec{\nabla}\left(-\frac{G_{\text{N}}Mm}{r}\right), \tag{0.0.24}$$

and therefore the potential energy is

$$U = -\frac{G_{\text{N}}Mm}{r} + \text{constant}. \tag{0.0.25}$$

Spring In one dimension, if x is the coordinate displacement measured from the location where the spring is neither stretched nor compressed and $\widehat{i}$ is the associated unit vector, the spring force is

$$\vec{F} = -kx\widehat{i} = -\partial_x\left(\frac{1}{2}kx^2\right)\widehat{i}; \tag{0.0.26}$$

the interpretation here is that it pushes when the spring is compressed ($x < 0$); whereas it pulls when the spring is stretched ($x > 0$). The potential energy is therefore

$$U = \frac{1}{2}kx^2 + \text{constant}. \tag{0.0.27}$$

Friction Friction is a phenomenological macroscopic force law that arises from the microscopic interactions between two rough surfaces. Its key property is that it opposes the motion;

i.e., opposite in direction to the velocity at a given instant. One form of friction is $-f\vec{v}$, where $f > 0$ and $\vec{v} = \mathrm{d}\vec{x}/\mathrm{d}t$ is the velocity of the body in question. Note that the work done by such a force is

$$W_f = -f \int_A^B \vec{v} \cdot \mathrm{d}\vec{x} = -f \int_A^B \vec{v}^2 \mathrm{d}t \leq 0. \tag{0.0.28}$$

Since this integral involves $\vec{v}^2$, a strictly non-negative quantity, this means it can be zero only when $\vec{v}$ is zero along the entire path–an impossibility. In turn, the work done by friction when the body returns to the same point (i.e., $A = B$) cannot be zero. Now, if it were possible to write the friction force as a negative gradient of a potential – i.e., if friction were actually conservative – we have seen that work done for $A \to A$ would be zero because we would be taking the difference of the potential energy at the same point. Hence, friction cannot be conservative.