Relativity

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Contents

1 Galilean Relativity

The core tenet of relativity is that physical laws must be indistinguishable – the fundamental equations of physics must take the same form – when we switch from one inertial frame to another. The difference between Galilean Relativity underlying Newton's laws of motion and Einstein's Special Relativity is the very different notions of an inertial frame. Let us begin with the former, since it is likely already familiar.

Newton's 1st law of motion is really a *definition* of an inertial frame: if there are no forces acting on an arbitrary system and if *all* such force-free systems travel at constant velocities, then the frame in which these observations are made is an inertial one.

Newton's 2nd law In such an inertial frame, using Cartesian coordinates $\vec{x}$ to describe the location os some mass m, its acceleration is governed by Newton's second law:

$$m\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} \equiv m\ddot{\vec{x}} = \vec{F}_{\text{total}}, \tag{1.0.1}$$

where $\vec{F}_{\text{total}}$ denotes the total force acting on it. Note, however, that Newton's 2nd law does not tell you what the forces $\{\vec{F}\}$ are; they are to be determined by empirical observation of the real world.

Flat Space In writing eq. (1.0.1) using Cartesian coordinates, there is an implicit assumption that Newton's laws of motion applies in *flat space*, whose precise definition I shall delay for a while. Roughly speaking flat space is where the rules of Euclidean geometry holds: parallel lines do not cross, sum of internal angles of a triangle equals π, the Pythagorean theorem holds, etc. For example, the surface of a perfectly spherical ball is *not* a flat space; parallel lines can meet, and sum of the internal angles of a triangle is not necessarily π. Being in flat space means, we may extend a straight line from mass m_1 at $\vec{x}_1$ and mass m_2 at $\vec{x}_2$, and denote the resulting vector as

$$\vec{\Delta}_{1\to 2} \equiv \vec{x}_2 - \vec{x}_1. \tag{1.0.2}$$

In particular, if both masses experience no external forces, $\ddot{\vec{x}}_{1,2} = 0$, we must also have

$$\ddot{\vec{\Delta}}_{1\to 2} = 0, \tag{1.0.3}$$

whose solution tells us the relative displacement between them must amount to constant velocity motion:

$$\vec{x}_2(t) - \vec{x}_1(t) = \vec{\Delta}_{1\to 2} = \vec{\Delta}_0 + \vec{V}\cdot t, \tag{1.0.4}$$

for time ($t-$)independent 'initial displacement' $\vec{\Delta}_0$ and 'initial velocity' $\vec{V}$.

Problem 1.1. Force-Free Parallel Lines on 2-Sphere Consider two masses m_1 and m_2 located on the unit $2-$sphere, with trajectories $\vec{y}_1(t)$ and $\vec{y}_2(t)$. We shall let their initial velocities at $t = 0$ be perpendicular to the equator at $\theta = \pi/2$. This means they are initially parallel. Below, we shall verify that the following trajectories are indeed force-free:

$$y^i_{1,2} = \left(\frac{\pi}{2} - v_0 t, \phi_{1,2}\right), \tag{1.0.5}$$

where $\phi_{1,2}$ are the constant azimuthal angles of the masses' motion.

Verify that the velocities of $m_{1,2}$ are both $-v_0\widehat{\theta}$. What time t do they meet at the North Pole? For $\Delta\phi \equiv \phi_2 - \phi_1$ small enough – namely, the trajectories are nearby enough – so that the local region of space containing the two masses at a given time t can be considered nearly flat space, show that the displacement vector joining m_1 to m_2 is

$$\vec{\Delta}_{1\to 2} = \cos(v_0 t)\Delta\phi \cdot \widehat{\phi}. \tag{1.0.6}$$

This problem illustrates the difference between force-free motion on a curved space versus that in flat space: not only can initially parallel trajectories eventually meet, their relative displacements are not acceleration free – unlike their counterparts in flat space in equations (1.0.3) and (1.0.4). □

Recall that, if $\vec{X}$ is the relative displacement from point A to B, the Pythagorean theorem informs us that the square of the distance between them is $\vec{X}^2 \equiv \vec{X}\cdot\vec{X}$, where the $\cdot$ is the the ordinary dot product. In an infinitesimal region of space, the infinitesimal distance $\mathrm{d}\ell$ between $\vec{x}$ and $\vec{x}+\mathrm{d}\vec{x}$ is therefore

$$\mathrm{d}\ell^2 = \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}. \tag{1.0.7}$$

A word on notation: instead of labeling the Cartesian components $\{x, y, z\}$, we shall instead call them $\{x^1, x^2, x^3\}$. Here, $\{x^i | i = 1, 2, 3\}$ does not mean x raised to the ith power; but rather the ith component of the Cartesian coordinate vector $\vec{x}$. The Pythagorean theorem reads, in $3D$ space,

$$\mathrm{d}\ell^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2. \tag{1.0.8}$$

This is also a good place to introduce Einstein summation notation. First, we define the Kronecker delta,

$$\delta_{ij} = 1 \qquad \text{if } i = j \tag{1.0.9}$$

$$= 0 \qquad \text{if } i \neq j. \tag{1.0.10}$$

Notice, this is simply the identity matrix in index notation. Then, decree that

> whenever a pair of indices are repeated – for e.g., $A^i B^i$ – they are implicitly summed over; namely, $A^i B^i \equiv \sum_i A^i B^i$.

For instance, the dot product between $\vec{a}$ and $\vec{b}$ is now expressible as

$$\vec{a}\cdot\vec{b} = \sum_i a^i b^j = \delta_{ij} a^i b^j. \tag{1.0.11}$$

We may thus rephrase the Pythagorean theorem as

$$\mathrm{d}\ell^2 = \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j. \tag{1.0.12}$$

If we choose instead some other (possibly curvilinear) coordinates $\{y^i\}$, we may simply compute the Jacobian $\partial x^i/\partial y^a$ in order to obtain $\mathrm{d}\ell$ in this new system:

$$\mathrm{d}\ell^2 = \delta_{ij}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}\mathrm{d}y^a\mathrm{d}y^b = \left(\frac{\partial\vec{x}}{\partial y^a}\cdot\frac{\partial\vec{x}}{\partial y^b}\right)\mathrm{d}y^a\mathrm{d}y^b \equiv g_{a'b'}(\vec{y})\mathrm{d}y^a\mathrm{d}y^b. \tag{1.0.13}$$

Much of vector calculus operations follows from this object $g_{a'b'}$, usually dubbed the 'metric tensor'.

Problem 1.2. Extremal length implies straight lines Let $\vec{z}(\lambda_0 \leq \lambda \leq \lambda_1)$ be a path in space that joins $\vec{x}'$ to $\vec{x}$:

$$\vec{z}(\lambda = \lambda_0) = \vec{x}', \qquad \vec{z}(\lambda = \lambda_1) = \vec{x}. \tag{1.0.14}$$

Total length of this path can be defined as

$$\ell(\vec{x}' \leftrightarrow \vec{x}) = \int_{\vec{x}'}^{\vec{x}} \sqrt{\mathrm{d}\vec{z} \cdot \mathrm{d}\vec{z}} = \int_{\lambda_0}^{\lambda_1} \mathrm{d}\lambda \sqrt{(\mathrm{d}\vec{z}/\mathrm{d}\lambda)^2}, \tag{1.0.15}$$

where $(\mathrm{d}\vec{z}/\mathrm{d}\lambda)^2 \equiv (\mathrm{d}\vec{z}/\mathrm{d}\lambda) \cdot (\mathrm{d}\vec{z}/\mathrm{d}\lambda)$. If ℓ is extremized, show that $\vec{z}$ are straight lines joining $\vec{x}'$ to $\vec{x}$; namely,

$$\vec{z} = \vec{x}' + f(\lambda)(\vec{x} - \vec{x}'); \tag{1.0.16}$$

where f is an arbitrary but monotonically increasing function of λ subject to the boundary conditions $f(\lambda = \lambda_0) = 0$ and $f(\lambda = \lambda_1) = 1$. □

Covariance Now, even though we defined Newton's second law eq. (1.0.1) using Cartesian coordinates, we may ask how to rephrase it in *arbitrary* ones. Afterall, a car or building should function in exactly the same way no matter what coordinate system the engineer used to design them. Geometrically, the length of a curve or the area of some 2D surface ought not depend on the coordinates used to parametrize them. Coordinates are important but merely technical intermediate tools to describe Nature herself. This demand that an equation of physics be expressible in arbitrary coordinate system – that the rules of calculation remains the same – is known as *covariance*.

We may begin with a function of space $f(\vec{x})$, which returns a unique number f given a unique location $\vec{x}$ in space – temperature of a medium at some point $\vec{x}$ is an example. Imagine the trajectory of a point mass $\vec{x}(t)$ passing through this medium, so that $f(\vec{x}(t))$ is the value f measured by it as a function of time. The time derivative is, by the chain rule,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\partial f}{\partial x^i}. \tag{1.0.17}$$

If we change the coordinate systems,

$$\vec{x} = \vec{x}(\vec{y}), \tag{1.0.18}$$
$$\vec{x}(t) \equiv \vec{x}(\vec{y}(t)); \tag{1.0.19}$$

then we had better get back the same f as long as $\vec{x} = \vec{x}(\vec{y})$ remains the same point:

$$f'(\vec{y}) \equiv f(\vec{x}(\vec{y})), \tag{1.0.20}$$

where the prime does not denote a derivative, but rather $f'(\vec{y})$ is the function f but now written in the $\vec{u}$ coordinate system. This, in fact, is the definition of a scalar function. Moreover, we may consider the time derivative

$$\frac{\mathrm{d}f'(\vec{y}(t))}{\mathrm{d}t} = \frac{\mathrm{d}y^a}{\mathrm{d}t}\frac{\partial f(\vec{x}(\vec{y}(t)))}{\partial y^a} = \frac{\mathrm{d}y^a}{\mathrm{d}t}\frac{\partial x^i}{\partial y^a}\frac{\partial f(\vec{x}(t))}{\partial x^i} = \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\partial f(\vec{x}(t))}{\partial x^i}. \tag{1.0.21}$$

Since f itself was arbitrary, we may therefore identify the *tangent vector* along the trajectory to be

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}y^a}{\mathrm{d}t}\frac{\partial}{\partial y^a} = \frac{\mathrm{d}y^a}{\mathrm{d}t}\frac{\partial x^i}{\partial y^a}\frac{\partial}{\partial x^i} = \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\partial}{\partial x^i}. \tag{1.0.22}$$

Notice, from the second and last equality, that this notion of a tangent vector – i.e., the velocity tangent to some prescribed path – takes the same form no matter the coordinate system used. The two expressions in the $\vec{x}-$ and $\vec{y}-$system are in fact related by a contraction with the relevant Jacobian. Moreover, notice $\mathrm{d}/\mathrm{d}t$ for an arbitrary trajectory is really a superposition of the partial derivatives with respect to the coordinates employed; hence the collection of all such $\mathrm{d}/\mathrm{d}t$ is the vector space at a given point in space spanned by these $\{\partial_i\}$.

Let's turn to the second derivative version of $\mathrm{d}/\mathrm{d}t$, which we shall denote as $D^2/\mathrm{d}t^2$. We demand, like $\mathrm{d}/\mathrm{d}t$, that it takes the same form no matter the coordinate system used. The answer is

$$\frac{D^2}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}^2 y^i}{\mathrm{d}t^2} + \Gamma^i{}_{ab}(\vec{y})\frac{\mathrm{d}y^a}{\mathrm{d}t}\frac{\mathrm{d}y^b}{\mathrm{d}t}\right)\frac{\partial}{\partial y^i}, \tag{1.0.23}$$

where the $\Gamma^i{}_{ab}$ are known as Christoffel symbols. For any spatial metric $\mathrm{d}\ell^2 = g_{ab}\mathrm{d}y^a\mathrm{d}y^b$, it can be computed as

$$\Gamma^i{}_{ab}(\vec{y}) = \frac{1}{2}(g^{-1})^{ic}(\vec{y})\left(\partial_{y^a}g_{bc} + \partial_{y^b}g_{ac} - \partial_{y^c}g_{ab}\right). \tag{1.0.24}$$

By viewing g_{ab} as a matrix (with a and b being the row and column number) we have defined $(g^{-1})^{ab}$ as the (a,b)-component of its inverse. For the flat metric at hand, if $\vec{x}$ are Cartesian coordinates and $\vec{y}$ are some other (possibly curvilinear) ones, so that $g_{ab} = \partial_{y^a}\vec{x}\cdot\partial_{y^b}\vec{x}$,

$$\begin{aligned}\Gamma^i{}_{ab}(\vec{y}) &= \frac{1}{2}(g^{-1})^{ic}(\vec{y})\left(\partial_{y^a}(\partial_{y^b}\vec{x}\cdot\partial_{y^c}\vec{x}) + \partial_{y^b}(\partial_{y^a}\vec{x}\cdot\partial_{y^c}\vec{x}) - \partial_{y^c}(\partial_{y^a}\vec{x}\cdot\partial_{y^b}\vec{x})\right) && (1.0.25)\\ &= \frac{1}{2}(g^{-1})^{ic}(\vec{y})\left(2\partial_{y^a y^b}\vec{x}\cdot\partial_{y^c}\vec{x} + \partial_{y^a}\vec{x}\cdot\partial_{y^b y^c}\vec{x} + \partial_{y^b}\vec{x}\cdot\partial_{y^a y^c}\vec{x} - \partial_{y^c y^a}\vec{x}\cdot\partial_{y^b}\vec{x} - \partial_{y^a}\vec{x}\cdot\partial_{y^c y^b}\vec{x}\right)\\ &= (g^{-1})^{ic}(\vec{y})\frac{\partial\vec{x}}{\partial y^a\partial y^b}\cdot\frac{\partial\vec{x}}{\partial y^c}. && (1.0.26)\end{aligned}$$

The inverse of the metric is

$$(g^{-1})^{ab}(\vec{y}) = \delta^{ij}\frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^j} \tag{1.0.27}$$

because

$$\begin{aligned}(g^{-1}g)^a{}_b = (g^{-1})^{ac}g_{cb} &= \frac{\partial y^a}{\partial x^i}\frac{\partial y^c}{\partial x^i}\frac{\partial x^l}{\partial y^c}\frac{\partial x^l}{\partial y^b} && (1.0.28)\\ &= \frac{\partial y^a}{\partial x^i}\frac{\partial x^l}{\partial x^i}\frac{\partial x^l}{\partial y^b} = \frac{\partial y^a}{\partial x^i}\delta^l_i\frac{\partial x^l}{\partial y^b} && (1.0.29)\\ &= \frac{\partial y^a}{\partial x^l}\frac{\partial x^l}{\partial y^b} = \frac{\partial y^a}{\partial y^b} = \delta^a_b. && (1.0.30)\end{aligned}$$

In other words,

$$\frac{D^2}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}^2 y^i}{\mathrm{d}t^2} + (g^{-1})^{ic}(\vec{y})\frac{\partial \vec{x}}{\partial y^a \partial y^b} \cdot \frac{\partial \vec{x}}{\partial y^c}\frac{\mathrm{d}y^a}{\mathrm{d}t}\frac{\mathrm{d}y^b}{\mathrm{d}t}\right)\frac{\partial}{\partial y^i} \tag{1.0.31}$$

$$= \left(\frac{\mathrm{d}^2 y^i}{\mathrm{d}t^2} + \frac{\partial y^i}{\partial x^l}\frac{\partial y^c}{\partial x^l}\frac{\partial x^k}{\partial y^a \partial y^b}\frac{\partial x^k}{\partial y^c}\frac{\mathrm{d}y^a}{\mathrm{d}t}\frac{\mathrm{d}y^b}{\mathrm{d}t}\right)\frac{\partial}{\partial y^i} \tag{1.0.32}$$

$$= \left(\frac{\mathrm{d}^2 y^i}{\mathrm{d}t^2} + \frac{\partial y^i}{\partial x^l}\frac{\partial x^l}{\partial y^a \partial y^b}\frac{\mathrm{d}y^a}{\mathrm{d}t}\frac{\mathrm{d}y^b}{\mathrm{d}t}\right)\frac{\partial}{\partial y^i}. \tag{1.0.33}$$

Problem 1.3. Coordinate Transformation Consider changing coordinates $\vec{y} = \vec{y}(\vec{z})$, so that, for instance,

$$\frac{\partial}{\partial y^a} = \frac{\partial z^k}{\partial y^a}\frac{\partial}{\partial z^k} \qquad \text{and} \qquad \frac{\mathrm{d}y^i}{\mathrm{d}t} = \frac{\partial y^i}{\partial z^a}\frac{\mathrm{d}z^a}{\mathrm{d}t} \tag{1.0.34}$$

– show that $D^2/\mathrm{d}t^2$ does indeed take the same form:

$$\frac{D^2}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}^2 z^i}{\mathrm{d}t^2} + \frac{\partial z^i}{\partial x^l}\frac{\partial x^l}{\partial z^a \partial z^b}\frac{\mathrm{d}z^a}{\mathrm{d}t}\frac{\mathrm{d}z^b}{\mathrm{d}t}\right)\frac{\partial}{\partial z^i}. \tag{1.0.35}$$

Observe that, we recover the ordinary acceleration $\mathrm{d}^2 x^i/\mathrm{d}t^2$ when $\vec{z} = \vec{x}$; in fact, one approach to this problem is to show that

$$\ddot{x}^i \partial_{x^i} = \frac{D^2 z^i}{\mathrm{d}t^2}\partial_{z^i} \tag{1.0.36}$$

for arbitrary but invertible $\vec{x}(\vec{z})$. Moreover, since this definition takes the same form under arbitrary coordinate systems, we may take it to denote the fully covariant form of acceleration $a^i \equiv \ddot{z}^i + (\partial z^i/\partial x^l)(\partial^2 x^l/\partial z^a \partial z^b)\dot{z}^a \dot{z}^b$. □

Problem 1.4. Classical Mechanics on 2-Sphere We may exploit the first line in eq. (1.0.31) to describe acceleration $D^2 z^i/\mathrm{d}t^2$ on the 2−sphere. Simply view $\vec{x}$ as the unit-length Cartesian displacement vector parametrized in spherical coordinates:

$$\vec{x} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,. \tag{1.0.37}$$

Next, define

$$y^i \equiv (\theta, \phi). \tag{1.0.38}$$

Show that

$$g_{ab}(\vec{y}) = \frac{\partial \vec{x}}{\partial y^a} \cdot \frac{\partial \vec{x}}{\partial y^b} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix} \qquad \text{and} \qquad (g^{-1})^{ab}(\vec{y}) = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sin^2\theta \end{bmatrix}. \tag{1.0.39}$$

By computing $\partial y/\partial x$ and $\partial^2 x/\partial x \partial x$, show that the trajectories in eq. (1.0.5) are indeed acceleration-free: $D^2\theta/\mathrm{d}t^2 = 0 = D^2\phi/\mathrm{d}t^2$. □

Euclidean symmetry We record here without proof that the most general coordinate transformation $\vec{x} = \vec{x}(\vec{x}')$ that preserves the form of the metric in equations (1.0.7) and (1.0.12) – namely,

$$\mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x} = \left(\frac{\partial \vec{x}}{\partial x'^a} \cdot \frac{\partial \vec{x}}{\partial x'^b} \right) \mathrm{d}x'^a \mathrm{d}x'^b = \mathrm{d}\vec{x}' \cdot \mathrm{d}\vec{x}' \tag{1.0.40}$$

is given by

$$\vec{x} = \widehat{R} \cdot \vec{x}' + \vec{a}, \tag{1.0.41}$$

where $\widehat{R}$ is an orthogonal matrix obeying $\widehat{R}^{\mathrm{T}} \widehat{R} = \mathbb{I}$ and $\vec{a}$ is a constant vector. **YZ: Switch to Analytical Methods.**

Galiean Transformation and Newtonian Gravity An important example is that of Newtonian gravity of N point masses. In an inertial frame, Newton's second law for the A-th mass reads

$$m_A \ddot{\vec{x}}_A = -\sum_{B \neq A} \frac{G_{\mathrm{N}} m_A m_B (\vec{x}_A - \vec{x}_B)}{|\vec{x}_A - \vec{x}_B|^3}. \tag{1.0.42}$$

That we are in flat space is what allows us to write the displacement vector between m_A and m_B as $\vec{x}_A - \vec{x}_B$ and the associated distance as $|\vec{x}_A - \vec{x}_B|$. Additionally, let us perform the Galilean transformation

$$\vec{x} = \widehat{R} \cdot \vec{x}' + \vec{a} + \vec{V} \cdot t' \qquad \text{and} \qquad t = t', \tag{1.0.43}$$

where $\widehat{R}$, $\vec{V}$, and $\vec{a}$ are constant; moreover $\widehat{R}^{\mathrm{T}} \widehat{R} = \mathbb{I}$. This relates two inertial frames by a constant velocity displacement as well as spatial rotation and/or parity flips. Note that eq. (1.0.41) is a subset of eq. (1.0.43); i.e., where $\vec{V} = \vec{0}$.

For Newtonian gravity, eq. (1.0.43) leads to

$$\vec{x}_A - \vec{x}_B = \widehat{R} \cdot (\vec{x}'_A - \vec{x}'_B), \tag{1.0.44}$$

$$|\vec{x}_A - \vec{x}_B| = |\vec{x}'_A - \vec{x}'_B|, \tag{1.0.45}$$

whereas

$$\ddot{\vec{x}} = \widehat{R} \cdot \ddot{\vec{x}}'. \tag{1.0.46}$$

Altogether, Newtonian gravity now reads

$$m_A \widehat{R} \cdot \ddot{\vec{x}}'_A = -\sum_{B \neq A} \frac{G_{\mathrm{N}} m_A m_B \widehat{R} \cdot (\vec{x}'_A - \vec{x}'_B)}{|\vec{x}'_A - \vec{x}'_B|^3}. \tag{1.0.47}$$

In index notation,

$$m_A \widehat{R}^i{}_j \cdot \ddot{x}'^j_A = -\widehat{R}^i{}_j \sum_{B \neq A} \frac{G_{\mathrm{N}} m_A m_B (x'^j_A - x'^j_B)}{|\vec{x}'_A - \vec{x}'_B|^3}. \tag{1.0.48}$$

Multiplying both sides by $\widehat{R}^{\mathrm{T}}$, we see that Newtonian gravity is in fact invariant under the transformations in eq. (1.0.43):

$$m_A \ddot{\vec{x}}'_A = -\sum_{B\neq A} \frac{G_{\mathrm{N}} m_A m_B \cdot (\vec{x}'_A - \vec{x}'_B)}{|\vec{x}'_A - \vec{x}'_B|^3}. \tag{1.0.49}$$

To ensure that Newtonian gravity may be expressed in arbitrary coordinate systems, we contract both sides with the partial derivatives:

$$m_A \ddot{x}^i_A \partial_{x^i} = -\sum_{B\neq A} \frac{G_{\mathrm{N}} m_A m_B \cdot (x^i_A - x^i_B)}{|\vec{x}_A - \vec{x}_B|^3} \partial_{x^i}. \tag{1.0.50}$$

Previously, we have already seen that if $\vec{x}(\vec{z})$ is given the coordinate invariant version of the LHS is $\ddot{x}^i \partial_{x^i} = (D^2 z^i/\mathrm{d}t^2)\partial_{z^i}$. On the other hand, $\partial_{x^i} = (\partial z^a/\partial x^i)\partial_{z^a}$. We therefore arrive at

$$m_A \frac{D^2 z^a_A}{\mathrm{d}t^2} \partial_{z^a} = -\sum_{B\neq A} \frac{G_{\mathrm{N}} m_A m_B \cdot (x^i_A - x^i_B)}{|\vec{x}_A - \vec{x}_B|^3} \frac{\partial z^a}{\partial x^i} \partial_{z^a}. \tag{1.0.51}$$

Problem 1.5. 2-Body Newtonian Gravity: Spherical Coordinates Suppose a small mass m is orbiting a much heavier one M, i.e., $m \lll M$, so that Newton's law of gravity reduces for the small mass' trajectory $\vec{x}$ to

$$\ddot{x}^i = -\frac{G_{\mathrm{N}} M}{|\vec{x}|^2} \frac{\vec{x}}{|\vec{x}|}, \tag{1.0.52}$$

where $\vec{x}$ are Cartesian coordinates. Use eq. (1.0.35) to show that, in spherical coordinates,

$$\ddot{r} - r \cdot \dot{\theta}^2 - r \cdot \sin^2\theta \dot{\phi}^2 = -\frac{G_{\mathrm{N}} M}{r^2}, \tag{1.0.53}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{1.0.54}$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0. \tag{1.0.55}$$

For practical purposes, it is useful to choose the coordinate system such that the orbit takes place on the $\theta = \pi/2$ plane. □

Classical Mechanics & Galilean Symmetry For slowing moving classical systems ($v/c \ll 1$), the Galilean transformation in eq. (1.0.43) are expected to preserve the form of all fundamental physical laws.

Mathematically, we may package it as the following matrix relation:

$$\begin{bmatrix} t \\ \vec{x} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \vec{0}^{\mathrm{T}} & 0 \\ \vec{V} & \widehat{R} & \vec{a} \\ 0 & \vec{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} t' \\ \vec{x}' \\ 1 \end{bmatrix}. \tag{1.0.56}$$

(The final row does not contain physical information; it is inserted just to make the matrix multiplication work out properly.) We see that a Galilean transformation can be encoded with a $(D+2)\times(D+2)$ matrix, containing the constant velocity $\vec{V}$, the rotation and/or parity flip $\widehat{R}$, and the constant spatial displacement $\vec{a}$. Denoting

$$\Pi\left(\vec{V},\widehat{R},\vec{a}\right) \equiv \left[\begin{array}{ccc} 1 & \vec{0}^{\mathrm{T}} & 0 \\ \vec{V} & \widehat{R} & \vec{a} \\ 0 & \vec{0}^{\mathrm{T}} & 1 \end{array}\right], \tag{1.0.57}$$

we multiply two such matrices to uncover

$$\Pi\left(\vec{V}_1,\widehat{R}_1,\vec{a}_1\right)\cdot\Pi\left(\vec{V}_2,\widehat{R}_2,\vec{a}_2\right) = \left[\begin{array}{ccc} 1 & \vec{0}^{\mathrm{T}} & 0 \\ \vec{V}_1 & \widehat{R}_1 & \vec{a}_1 \\ 0 & \vec{0}^{\mathrm{T}} & 1 \end{array}\right]\left[\begin{array}{ccc} 1 & \vec{0}^{\mathrm{T}} & 0 \\ \vec{V}_2 & \widehat{R}_2 & \vec{a}_2 \\ 0 & \vec{0}^{\mathrm{T}} & 1 \end{array}\right] \tag{1.0.58}$$

$$= \left[\begin{array}{ccc} 1 & \vec{0}^{\mathrm{T}} & 0 \\ \vec{V}_1+\widehat{R}_1\vec{V}_2 & \widehat{R}_1\widehat{R}_2 & \widehat{R}_1\vec{a}_2+\vec{a}_1 \\ 0 & \vec{0}^{\mathrm{T}} & 1 \end{array}\right] \tag{1.0.59}$$

$$= \Pi\left(\vec{V}_1+\widehat{R}_1\vec{V}_2,\widehat{R}_1\widehat{R}_2,\widehat{R}_1\vec{a}_2+\vec{a}_1\right). \tag{1.0.60}$$

The identity transformation is

$$\mathbb{I}_{(D+2)\times(D+2)} = \Pi\left(\vec{0},\mathbb{I}_{D\times D},\vec{0}\right). \tag{1.0.61}$$

and therefore

$$\Pi\left(\vec{V},\widehat{R},\vec{a}\right)^{-1} = \Pi\left(-\widehat{R}^{\mathrm{T}}\vec{V},\widehat{R}^{\mathrm{T}},-\widehat{R}^{\mathrm{T}}\vec{a}\right). \tag{1.0.62}$$

These relations verify that $\{\Pi(\vec{V},\widehat{R},\vec{a})\}$ forms a group.

Problem 1.6. Derivatives Explain why

$$\left[\begin{array}{cc} \frac{\partial t}{\partial t'} & \frac{\partial t}{\partial x'^i} \\ \frac{\partial x^a}{\partial t'} & \frac{\partial x^a}{\partial x'^b} \end{array}\right]^{-1} = \left[\begin{array}{cc} \frac{\partial t'}{\partial t} & \frac{\partial t'}{\partial x^i} \\ \frac{\partial x'^a}{\partial t} & \frac{\partial x'^a}{\partial x^b} \end{array}\right]. \tag{1.0.63}$$

Use this result or otherwise to deduce from eq. (1.0.43) the relations

$$\partial_t = \partial_{t'} - V^a\widehat{R}^{ab}\partial_{x'^b} \tag{1.0.64}$$

and

$$\partial_{x^i} = \widehat{R}^{ij}\partial_{x'^j}. \tag{1.0.65}$$

These results are important in determining if certain partial differential equations of physics are in fact invariant under the Galilean transformations of eq. (1.0.43). □

Problem 1.7. Covariant Acceleration If the Cartesian $\vec{x}$ are given a transformation into $\vec{z}-$coordinates, i.e., $\vec{x}(\vec{z})$ is given, show that eq. (1.0.43) then implies

$$\frac{\partial x^a}{\partial z^i} = \widehat{R}^{ab}\frac{\partial x'^b}{\partial z^i} \tag{1.0.66}$$

and

$$\frac{\partial z^i}{\partial x^a} = \frac{\partial z^i}{\partial x'^b}\widehat{R}^{ab}. \tag{1.0.67}$$

Next, prove that the acceleration in eq. (1.0.35) is in fact invariant under Galilean transformations. $\square$

Surface Waves: Toy Model Let x^3 be the height of the 2D surface of some substance made out of many point particles – say, a rubber sheet. Let there be a wave propagating along the positive 1−direction, so that

$$x^3 = A\sin(x^1 - vt), \tag{1.0.68}$$

where A is the amplitude of the wave and v is its (constant) speed. These $\vec{x} = (x^1, x^2, x^3)$ are defined with respect to the rest frame of this substance. Now, if Galilean symmetry holds (cf. (1.0.43)), then in the inertial $\vec{x}'$−frame moving at velocity $\vec{V}$ parallel to the rubber sheet, namely

$$(x^1, x^2, x^3) = (x'^1 + a^1 + V^1 \cdot t, x'^2 + a^2 + V^2 \cdot t, x'^3), \tag{1.0.69}$$

we have

$$x'^3 = A\sin(x'^1 + a^1 + (V^1 - v)t). \tag{1.0.70}$$

In other words, the velocity of the wave in this new $\vec{x}'$−frame is now $V^1 - v$. We shall see the electromagnetic waves *do not* transform in such a manner.

2 Last update: February 23, 2025

References